UDC 539.376

NONLINEAR GOVERNING EQUATION OF A HEREDITARY MEDIUM AND METHODOLOGY OF DETERMINING ITS PARAMETERS

PMM, Vol.42, No.6, 1978, pp. 1107-1114

A. E. OSOKIN and Iu. V. SUVOROVA
(Moscow)

(Received November 14, 1977)

There is given a foundation for the model of a hereditary medium [1] based on the Rabotnov equations. There is poroposed a new method of determining the parameters of the equation, which affords the possibility of processing the experimental data obtained for different stress laws.

The general form of the governing equations under consideration is the following:

$$\varphi[\varepsilon(t)] = \sigma(t) + \int_0^t F(t-\tau)\sigma(\tau)\,d\tau \tag{0.1}$$

Here σ is the stress, ε is the strain, $\sigma = \varphi(\varepsilon)$ is the equation of the instantaneous strain curves, and $F(t-\tau)$ is the kernel of the integral equation governing the hereditary properties of the material.

A method of determining the parameters of a linear equation of hereditary type with an exponential fractional kernel of Rabotnov according to the creep curve was developed earlier [2], i. e., a particular case of (0.1) for $\varphi(\varepsilon) = E\varepsilon$ is considered. This method can be used only for linear equations and experiments with constant stresses when the $\sigma(\tau)$ under the integral sign does not vary with time. In this case, tables of the exponential-fractional Rabotnov functions can be used efficiently in the computations [3, 4]. It is impossible to use integral transform methods in determining the parameters of a nonlinear model, hence, values of the parameters of the kernel, obtained along linear sections of the isochronic creep curves were used in [5]. The instantaneous curve had already been constructed according to known parameters.

The method proposed here to determine the parameters does not require application of integral transforms and is hence suitable for any equations of hereditary type with an arbitrary loading law.

1. Method of the computations of the governing equation parameters. Let us assume that K experiments have been conducted with quasistatic loading of a rod made of material being investigated in a given loading mode $\sigma = \sigma(t)$. Let us identify these experiments by the subscript k $(k = 1, 2, 3, \ldots, K)$; the strain mode in each experiment is determined by its dependence $\sigma_k(t)$, where all the experiments are conducted so that $d\sigma_k(t)/dt \geqslant 0$, i.e., unloading does not occur.

Let ε_{kr} denote the value of the strain in the k-th experiment at the time t_r and let us consider the times at which the strains are measured to be the same in all the experiments. The method which will be constructed below can be used, without substantial variation, even in the case when the strains in each of the experiments are obtained at arbitrarily selected times.

Let the experimental results be reduced to K tables governing the values of the strain at the times t_r $(1 \leqslant r \leqslant R)$. We assume that the kernel $F(t)$ is a continuous function of the time defined by a set of two (f, α) or three (f, α, β) parameters (the use of more than three parameters is inexpedient), which we shall denote by the single letter p; hence $p = p_*$ means that $f = f_*, \alpha = \alpha_*$ or $f = f_*$, $\alpha = \alpha_*$, $\beta = \beta_*$.

For a given p values of $\varphi(\varepsilon)$ can evidently be found at points ε_{kr}

$$\varphi_{kr} = \varphi[\varepsilon_{kr}(t_r)] = \sigma_k(t_r) + \int_0^{t_r} F(\tau, p)\, \sigma_k(t_r - \tau)\, d\tau \tag{1.1}$$

Here we used the identity

$$\int_0^t F(t-\tau)\, \sigma_k(\tau)\, d\tau \equiv \int_0^t F(\tau)\, \sigma_k(t-\tau)\, d\tau$$

which expresses the commutativity of the convolution of the functions $F(t)$ and $\sigma(t)$. The values of $\varphi(\varepsilon)$ at the points ε_{kr} are denoted in (1.1) by φ_{kr} analogously to the strains. Let us introduce the continuous functions $\varphi_k(\varepsilon)$ which are obtained by linear interpolation in the values of φ_{kr}

$$\varphi_k(\varepsilon) = [\varphi_{kr}(\varepsilon_{k(r+1)} - \varepsilon) + \varphi_{k(r+1)}(\varepsilon - \varepsilon_{kr})] \,/\, [\varepsilon_{k(r+1)} - \varepsilon_{kr}]$$

Then the function $\varphi(\varepsilon)$ obtained by continuous "joining" of the functions $\varphi_k(\varepsilon)$ has the form

$$\varphi(\varepsilon) = \begin{cases} \varphi_k(\varepsilon), \quad \varepsilon_{k1} \leqslant \varepsilon \leqslant \varepsilon_{kR}, \quad 1 \leqslant k \leqslant K \\ \dfrac{\varphi_{kR}(\varepsilon_{(k+1)r_m} - \varepsilon) + \varphi_{(k+1)r_m}(\varepsilon - \varepsilon_{kR})}{\varepsilon_{(k+1)r_m} - \varepsilon_{kR}}, \quad \varepsilon_{kR} \leqslant \varepsilon \leqslant \varepsilon_{(k+1)r_m} \end{cases}$$

where r_m is the largest of the subscripts r such that $\varepsilon_{kR} \leqslant \varepsilon_{(k+1)r}$. If the experiments are conducted so that the segments $\varepsilon_{k1} \leqslant \varepsilon \leqslant \varepsilon_{kR}$ do not intersect, then we obtain $r_m = 1$ for all k. The construction of $\varphi(\varepsilon)$ by means of $\varphi_k(\varepsilon)$ is shown in Fig. 1.

To determine $\varphi(\varepsilon)$ and p which best describe the material under investigation, let us seek the set of parameters governing the nucleus for which the sections of the instantaneous stress-strain diagram corresponding to the early strain modes form a single smooth curve. The application of numerical methods for such a search, which will admit realization on an electronic computer, first requires quantitative formulation of the condition for the optimal selection of p. In order to obtain this condition and to reduce the problem of determining the parameters of the nucleus to a formal problem on the search for the minimum of a function of several variables (the role of these variables will be played by parameters governing the form of the nucleus), we introduce the concept of the governing functional of the inverse problem (we call the problem of determining the necleus parameters the inverse problem).

Let us assume that there is a set of dependences $\varepsilon_k(t)$, $\sigma_k(t)$ of the material which is described exactly by the governing equation

$$\varphi_*(\varepsilon) = \sigma(t) + \int_0^t F(p_*, \tau)\sigma(t-\tau)\,d\tau \tag{1.2}$$

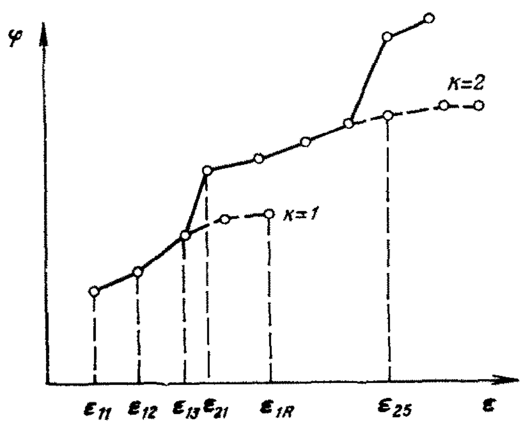

Fig. 1

and let us consider the set Φ of curves $\varphi(\varepsilon)$ obtained by the method examined above, when p takes on values from a certain set P. A mutually one-to-one correspondence can be set up between Φ and P, hence, any functional defined on Φ is, moreover a function of the argument p defined in P. So that by determining some functional on Φ, a function of the argument p whose domain of definition is the set P can be set in correspondence to this functional. We call this function the function corresponding to the functional. It evidently has two or three arguments, which are the parameters forming the set p.

Let us call the functional continuous if a continuous function corresponds to it. Let us call the continuous functional $\Delta(\varphi)$ whose domain of definition is the set Φ, governing if $\Delta(\varphi) \geqslant 0$ and $\Delta[\varphi(p))] = 0$ if and only if $p = p_*$.

We shall say that a single-parametric family of functionals $\Delta_N(\varphi)$ defined on Φ approximates the governing functional $\Delta(\varphi)$ as $N \to \infty$ if

$$\lim[\Delta_N(\varphi) - \Delta(\varphi)] = 0, \quad N \to \infty, \quad \varphi \in \Phi$$

The sense of introducing the governing functionals is clear from the definitions presented. In fact, if at least one such functional is constructed, it is sufficient to find a p such that $\Delta[\varphi(p)] = 0$, and the inverse problem is solved. The process of seeking the root of this equation is equivalent to searching for the minimum of the function corresponding to the given functional, i.e., the function of two or three variables. This problem has been studied well, and different methods have been developed for its solution (random search, formal search, method of gradients, etc.) which are realized in standard programs.

Before proceeding to the construction of the governing functionals, let us refine some experimental information which must be available for the solution of the inverse problem. Let us say that the experimental data are adequate to the determination of an instantaneous strain-stress diagram and of the parameters of the nucleus within the framework of a certain class of functions and a manifold of sets of parameters if these manifolds contain just one instantaneous diagram and just one set of parameters such that (1.2) is satisfied identically for all values of the strain and stress obtained in the experiment. This condition must be verified for each specific set of experimental data and given instantaneous diagrams and heredity nuclei.

Let us construct the governing functional in the case when the desired instantaneous curve $\varphi_*(\varepsilon)$ is a polynomial of degree N and the number R of points obtained at each loading satisfies the inequality $R \geqslant N+1$. For this we construct a Lagrange interpolation polynomial of degree N, which we denote by $L_{kN}(\varepsilon, p)$, where $\varepsilon_{11} \leqslant \varepsilon \leqslant \varepsilon_{KR}$ and we consider the following functionals

$$\Delta_{qN}(p) = \sum_{Q=1}^{q} \sum_{k=1}^{K-1} \max_{(\varepsilon)} \left| \frac{d^Q}{d\varepsilon^Q} [L_{kN}(\varepsilon, p) - L_{(k+1)N}(\varepsilon, p)] \right| \tag{1.3}$$

It follows from (1.3) that $\Delta_{qN}(p_*) = 0$ and the first condition imposed on the governing functional is satisfied. Let us show that $p = p_*$ follows from $\Delta_{qN}(p) = 0$ if the experimental data are adequate for the determination of the nucleus and the instantaneous strain-stress diagram. Indeed, it follows from $\Delta_{qN}(p) = 0$ that $L_{kN}(\varepsilon, p)$ is a polynomial of degree N and p is a set of parameters defining the nucleus such that (1.2) is satisfied identically for all the strains and stresses obtained in the experiments. Therefore $p = p_*$. Thus, the following theorem is proved: If a material satisfies the governing equation (1.2), and $\varphi_*(\varepsilon)$ is a polynomial of degree N, then each of the functionals given by (1.3) is a governing functional of the inverse problem.

It is known that the class of functions allowing an approximation in a finite interval by means of polynomials is sufficiently broad. The instantaneous strain-stress diagram is continuous at least, and therefore, by the Weierstrass theorem, can be approximated as accurately as required by a polynomial of degree N.

Let $\Delta_N(p)$ denote any of the functionals defined by (1.3). Then in the general case of a continuous instantaneous diagram, functionals of the form $\Delta_N(p)$ will allow the construction of a family of functionals which approximate the governing functional as $N \to \infty$. If such a one-parameter family of functionals is denoted by $\Delta_{*N}(p)$, then the governing functional in the general case of a continuous instantaneous diagram can be represented in the form $\Delta(p) = \lim_{N\to\infty} \Delta_{*N}(p)$.

For the initial confirmation of the method on an electronic computer, the parameters of a known governing equation were calculated, the results of the computation were compared with the exact values of the parameters and the error in the method was estimated for different governing functionals. The results of solving these test problems turned out to be best for the functional Δ_{1N} which was also used in calculations whose results are presented below. The formal search method was used to seek the minimum of this functional.

2. **Verification of the method in application to Rabotnov equation.** In order to clarify the possibility of applying the method to specific examples, it was first used to process the same data on the creep

of fiberglass which were the basis for constructing the nonlinearly hereditary model [5] and the same governing equation was selected

$$\varphi(\varepsilon) = \sigma(t) + f\int_0^t \partial_{-\alpha}(\beta, t-\tau)\,\sigma(\tau)\,d\tau \qquad (2.1)$$

$$\partial_{-\alpha}(\beta, t-\tau) = (t-\tau)^{-\alpha}\sum_{n=0}^{\infty}\frac{\beta^n (t-\tau)^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)]}$$

Here $\partial_{-\alpha}(\beta, t-\tau)$ is the Rabotnov function which is the kernel of the integral equation and governing the hereditary properties of the material. Equation (2.1) contains three parameters, f, α, β as well as the function $\varphi(\varepsilon)$.

In calculating φ_{kr} by means of (1.1) in the case of a kernel having the form of an exponential fractional function, it is necessary to evaluate the integral

$$I(t) = \int_0^t \partial_{-\alpha}(\beta, \tau)\,d\tau = \frac{t^{-\alpha}}{1-\alpha}\sum_{n=0}^{\infty}\frac{\beta^n t^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)](n+1)} \qquad (2.2)$$

For $x = \beta t^{(1-\alpha)} > 4$ it is convenient to evaluate $I(t)$ by starting from the asymptotic representation

$$I(t) = -\frac{1}{\beta} - t^{(1-\alpha)}\sum_{n=2}^{\infty}\frac{\beta^{-n} t^{-n(1-\alpha)}}{\Gamma[1+(1-\alpha)(1-n)]}$$

For $x < 1$ the series (2.2) converges rapidly and the evaluation of $I(t)$ also causes no difficulties. The greatest difficulties occur for $1 < x < 4$ since the rate of convergence of the series (2.2) drops abruptly. Let us examine one calculational recipe which permits reducing somewhat the volume of calculations needed. Let us introduce the function $J(\eta)$ connected with the series (2.2) by the relationship

$$J(\eta) = \sum_{n=0}^{\infty} a_n \eta^n, \quad a_n = \frac{t^{-\alpha} x^n}{(1-\alpha)(n+1)\Gamma[(1+n)(1-\alpha)]} \qquad (2.3)$$

$$\frac{dJ(\eta)}{d\eta} = \sum_{n=0}^{\infty} n a_n \eta^{(n-1)}$$

Since the series (2.3) converge uniformly (both series are majorized, the first by the series (2.2), and the second by the series defining $\partial_{-\alpha}(\beta, t)$), the function $J(\eta)$ is a differentiable function of the argument η, and therefore, admits of the representation

$$I(t) = J(1) = J(1-\delta) + \left[\frac{dJ(1-\delta)}{d\eta}\right]\delta + O(\delta^2), \quad \delta > 0$$

The series (2.3) evidently converge substantially better than (2.2). Electronic computer computations showed that the calculation using (2.3) yields not more than a

3% error for $\delta = 0.1$ if 100 terms are retained in (2.3).

Experimental creep curves (solid curves) underlying the computations [5] are presented in Fig. 2, whereupon the following values were obtained for the parameters: $\alpha = 0.885$, $\beta = 0.112\ \mathrm{min}^{-(1-\alpha)}$, $f = 0.116\ \mathrm{min}^{-(1-\alpha)}$. Stress values equal to $8.12 \cdot 10^5$, $12.18 \cdot 10^5$, $14.21 \cdot 10^5\ \mathrm{N/m^2}$ correspond to curves *1, 2, 3*. The instantaneous strain curve obtained is presented in Fig. 3 (curve 1), curve 2 is the diagram obtained in [5] to which the following parameters correspond: $\alpha = 0.8$, $\beta = 0.32\ \mathrm{min}^{-(1-\alpha)}$, $f = 0.26\ \mathrm{min}^{-(1-\alpha)}$.

The results obtained were then used to construct the creep curves. The dashed lines in Fig. 2 portray the results of computations of this paper, while the dash-dot lines

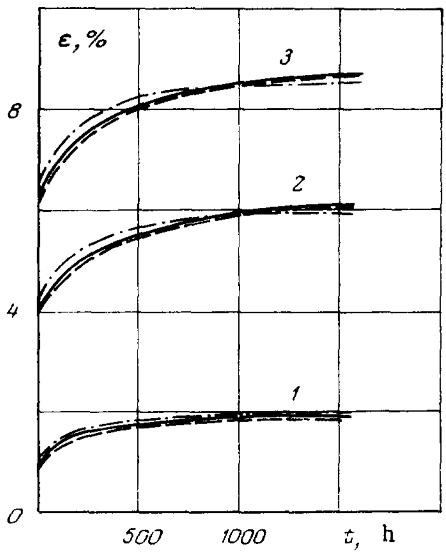

Fig. 2

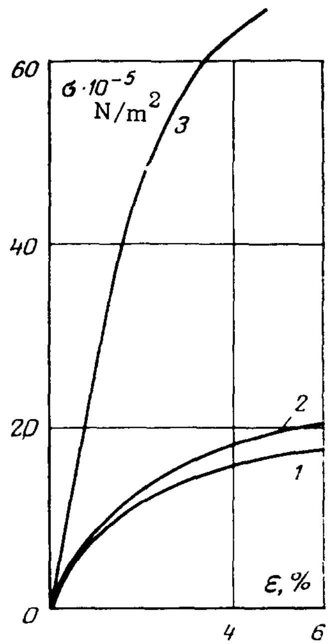

Fig. 3

are the results of [5]. As is seen, the method developed permits a more accurate computation of the parameters of the governing equation of the material so that the calculated creep curves agree almost completely with the experimental curves.

3. Foundation for the possibility of using the governing equation with an Abel kernel. As already mentioned above, the governing equation (0.1) with the Iu. N. Rabotnov kernel can be used sufficiently simply just for creep computations with constant stress. In order for the nonlinear hereditary governing equation to be used effectively for other modes also, it was proposed [1] to simplify the kernel of the integral equation and to take it in the form of the Abel kernel $F(t) = f t^{-\alpha}$, which is characterized by two parameters. Processing of a large quantity of experimental results confirmed the possibility of using it, the simplicity in this case being evident. Equation (0.1) can now be written in the form

$$\varphi(\varepsilon) = \sigma(t) + f(1-\alpha)\int_0^t (t-\tau)^{-\alpha}\sigma(\tau)\,d\tau \qquad (3.1)$$

The value of α is ordinarily taken in the interval $0 < \alpha < 1$ since for $\alpha = 0$ the kernel loses it singularity and becomes regular while for $\alpha = 1$ the singularity is not integrable. Insertion of the factor $1 - \alpha$ in front of the integral sign significantly extends the possibility of using (3. 1): if we put $\alpha = 1$, then the integral term vanishes. This permits combining two fundamental presently-existing models within the framework of a single approach. The first is the simplest, it is the ordinary deformation theory (the Karman-Taylor-Rakhmatulin scheme in dynamics, which assumes the existence of a singly dynamic strain diagram independent of the strain rate). Setting $\alpha = 1$ in (3. 1), we arrive at the governing equation $\sigma = \varphi(\varepsilon)$, hence the strain diagram is already independent of the loading rate and the prehistory of the process. If we set $\alpha = 0$ in (3. 1), we then obtain after differentiation with respect to time

$$\varphi'\varepsilon^{\cdot} = \sigma^{\cdot} + f\sigma \qquad (3.2)$$

If $\varphi' = E = \text{const}$, we then obtain the Maxwell model of a viscoelastic body. The second fundamental strain scheme, the Malvern — Sokolovskii model, is based on a dependence of the type (3. 2). Therefore, (3. 1) in the extreme cases $\alpha = 0$ and $\alpha = 1$ corresponds to two mutually-exclusive strain schemes, as mentioned earlier.

Equation (3. 1) was verified earlier in the processing of a large number of experiments for different loading modes in both metals and polymers and composites [1, 6 — 8]. In order to compare it with (2. 1) and to examine whether it does not result in a diminution in the number of parameters in the kernel (the exponential fractional kernel contains three parameters, and the Abel kernel only two), and in an increase in the errors in the computations, (3. 1) was applied to the same data on fiberglass creep which was spoken about above [5].

The method elucidated above in Sec. 1 was used in determining the parameters of the kernel; it hence turned out that $\alpha = 0.9593$ and $f = 3.27\ \text{min}^{-(1-\alpha)}$ for the Abel kernel. The appropriate instantaneous strain curves is presented in Fig. 3 (curve 3). It is seen that the instantaneous strain curves corresponding to (2. 1) and (3. 1) differ radically.

Furthermore, the parameters and $\varphi(\varepsilon)$ obtained were used to construct creep curves (see Fig. 2). The correspondence between computations and experiment was so good that the curves obtained could not be distinguished successfully in the scale of the sketch.

It therefore turns out that the simpler equation yields good results and can be recommended as a governing equation for nonlinear hereditary media.

The great advantage of (3. 1) is that it can be used for different loading modes. Hence, the procedure for determining the parameters, associated with the formulation of a purposeful experiment, is simplified substantially: it is sufficient to have two strain curves obtained at different loading rates. Strain diagrams of organic fibers obtained at the loading rates $\sigma^{\cdot} = 10^2\,\text{N/m}^2\cdot\text{s}$ and $\sigma^{\cdot} = 10^5\,\text{N/m}^2\cdot\text{s}$ are presented in Fig. 4. In the case of loading in the $\sigma^{\cdot} = \text{const}$ mode, (3. 1) results in the simple form

$$\varphi(\varepsilon) = \sigma^{\cdot} t + f\sigma^{\cdot} t^{(2-\alpha)} (2-\alpha)^{-1} \qquad (3.3)$$

The diagrams presented in Fig. 4 were used to determine the parameters by the method elucidated in Sect. 1. The following values were obtained: $\alpha = 0.9795$, $f = 1.507\ \mathrm{h}^{-(1-\alpha)}$. The instantaneous strain diagram $\sigma = \varphi(\varepsilon)$ is also presented in Fig. 4. The found parameters were then used to predict the behavior of the material under creep, i.e., for a loading in the $\sigma = \mathrm{const}$ mode. Results of the computations (dashed lines) and the experiment (solid lines) are presented in Fig. 5 for $\sigma = 2\cdot 10^5\ \mathrm{N/m^2}$ (curves 1) and $\sigma = 3.5\cdot 10^5\ \mathrm{N/m^2}$ (curves 2).

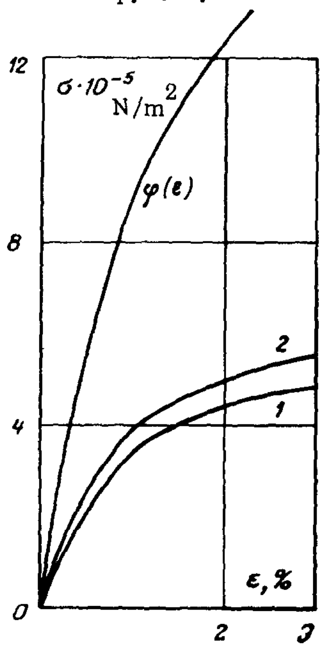

Fig. 4

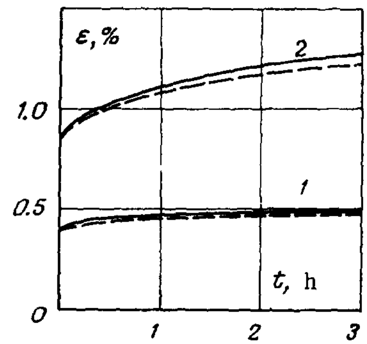

Fig. 5

In general, the experiment need not be performed absolutely in some definite loading mode to determine the parameters of (3.1). The modes $\sigma = \mathrm{const}$ and $\sigma^{\cdot} = \mathrm{const}$ were selected in [1] because the integral is easily taken in this case, (3.1) has a very simple form (3.3), and processing the experiments can be executed without using an electronic computer. However, the method developed in Sect. 1 to solve the inverse problems permits processing experiment data on an electronic computer for arbitrary loading modes.

A certain modification of (3.1), related to extraction of the elastic and plastic hereditary strain components, affords the possibility of using it to describe not only polymers and composites but also metals [6 — 8].

REFERENCES

1. Suvorova, Iu. V., Nonlinear effects in the deformation of hereditary media, Mekhan. Polimerov, No. 6, Moscow, 1977.

2. Zvonov, E. M., Malinin, N. I., Papernik, L. Kh., and Tseitlin, B. M., Determination of the creep characteristics of linear elastic hereditary materials by using a digital computer, Inzh. Zh., Mekhan. Tverd. Tela, No. 5, Moscow, 1968.

3. Rabotnov, Iu. N., Papernik, L. Kh., and Zvonov, E. N., Tables of the Exponential-fractional Functions of the Negative Parameters and of their Integrals. "Nauka", Moscow, 1969.

4. Rabotnov, Iu. N., Elements of the Hereditary Mechanics of Solids, "Nauka", Moscow, 1977.

5. Rabotnov, Iu. N., Papernik, L. Kh., and Stepanychev, E. I., Application on nonlinear hereditary theory to the description of time effects in polymer materials, Mekhan. Polimerov, No. 1, 1971.

6. Rabotnov, Iu. N. and Suvorova, Iu. V., On the law of metal strain under uniaxial loading, Izv. Akad. Nauk SSSR, No. 4, 1972.

7. Mel'shanov, A. F., Suvorova, Iu, V., and Khazanov, S. Iu., Experimental verification of the governing equation for metals under loading and unloading, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No. 6, 1974.

8. Suvorova, Iu. V., Taking account of the temperature in the hereditary theory of elastic-plastic media, Problemy Prochnosti, No. 2, 1977.

Translated by M. D. F.